

## Practice CS103 Final Exam II

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*We strongly recommend that you work through this exam under realistic conditions rather than just flipping through the problems and seeing what they look like. Setting aside three hours in a quiet space with your notes and making a good honest effort to solve all the problems is one of the single best things you can do to prepare for this exam. It will give you practice working under time pressure and give you an honest sense of where you stand and what you need to get some more practice with.*

This practice final exam is essentially the final exam from Fall 2015, with a few minor modifications (some of the problems we asked here got converted to problem set questions, so we replaced them with other exam questions) and others covered topics that have since be dropped from CS103 (namely, using self-reference to prove unrecognizability).

The exam is closed-book, closed-computer, limited note (one double-sided sheet of 8.5" × 11" paper decorated however you'd like).

You have three hours to complete this exam. There are 50 total points.

Question	Points	Graders
(1) Set Theory	/ 8	
(2) Logic and Relations	/ 6	
(3) Graph Theory	/ 12	
(4) Induction and Cardinality	/ 6	
(5) Regular and Context-Free Languages	/ 12	
(6) <b>R</b> and <b>RE</b> Languages	/ 6	

**Problem One: Set Theory****(5 Points)***(CS103 Midterm, Spring 2015)*

Recall that if  $S$  and  $T$  are sets, the difference of  $S$  and  $T$ , denoted  $S - T$ , is defined as the set of all elements that are in  $S$  but not in  $T$ .

- i. **(5 Points)** Are there any sets  $A$  and  $B$  such that  $\wp(A - B) = \wp(A) - \wp(B)$ ? If so, give an example of sets  $A$  and  $B$  with these properties. If not, prove why not.

- ii. **(3 Points)** Let  $A$  and  $B$  be arbitrary sets and consider the set  $S$  defined below:

$$S = \{ x \mid \neg(x \in A \rightarrow x \in B) \}$$

Write an expression for  $S$  in terms of  $A$  and  $B$  using the standard set operators (union, intersection, etc.), but without using set-builder notation. Briefly justify why your answer is correct.

**Problem Two: Logic and Relations****(6 Points)**

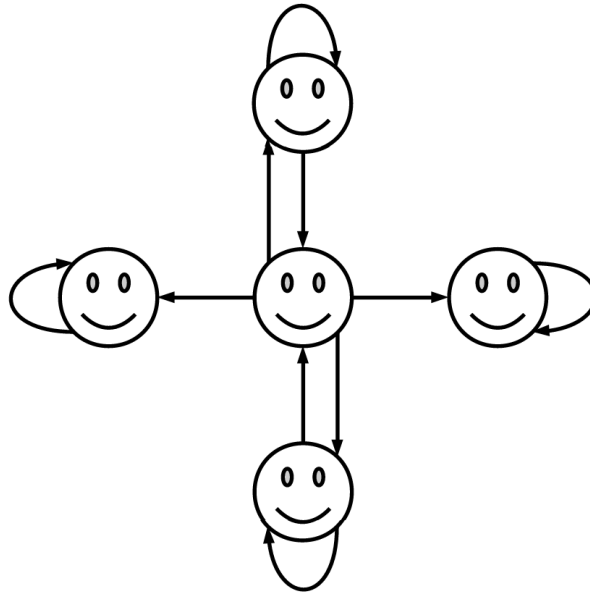
Suppose that you want to prove the implication  $P \rightarrow Q$ . Here are two possible routes you can take:

- Prove the implication by contradiction.
- Take the contrapositive of the implication, then prove the contrapositive by contradiction.

It turns out that these two proof approaches are completely equivalent to one another.

- i. **(2 Points)** State, in propositional logic, which statements you will end up assuming if you were to use each of the above proof approaches, then *briefly* explain why they're equivalent.

ii. (4 Points) Below is a drawing of a binary relation  $R$  over a set of people  $A$ :



For each of the following first-order logic statements about  $R$ , decide whether that statement is true or false. No justification is required, and there is no penalty for an incorrect guess.

1.  $\forall p \in A. \exists q \in A. pRq$

True

False

2.  $\exists p \in A. \forall q \in A. pRq$

True

False

3.  $\exists p \in A. (pRp \rightarrow \forall q \in A. qRq)$

True

False

4.  $\neg \forall p \in A. \forall q \in A. (p \neq q \rightarrow \exists r \in A. (pRr \wedge qRr))$

True

False

**Problem Three: Graph Theory****(12 Points)**

On Problem Set Four, you worked with graphs and played around with some of their properties. This question explores a new class of graphs and is designed to give you a chance to show us what you've learned along the way.

Let's begin with a definition. An undirected graph  $G = (V, E)$  is called a *friendship graph* if it satisfies the following requirement:

For any nodes  $u, v \in V$  where  $u \neq v$ ,

there is *exactly one* node  $z \in V$  where  $\{u, z\} \in E$  and  $\{v, z\} \in E$ .

As a reminder, undirected graphs cannot have edges from nodes back to themselves.

- i. **(4 Points)** Prove that if  $G = (V, E)$  is a friendship graph, then  $G$  does not contain any simple cycles of length four.

As a hint, draw pictures.

*(Extra space for your answer to Problem Three, Part (i), if you need it.)*

As a refresher from the previous page, an undirected graph  $G = (V, E)$  is called a **friendship graph** if it satisfies the following requirement:

For any nodes  $u, v \in V$  where  $u \neq v$ ,

there is **exactly one** node  $z \in V$  where  $\{u, z\} \in E$  and  $\{v, z\} \in E$ .

As a reminder, undirected graphs cannot have edges from nodes back to themselves.

- ii. **(5 Points)** Let  $G = (V, E)$  be a friendship graph with at least two nodes. Prove that for every node  $v \in V$ , there's a simple cycle of length three that contains  $v$ .

As a hint, draw pictures.



*(Extra space for your answer to Problem Three, Part (ii), if you need it.)*

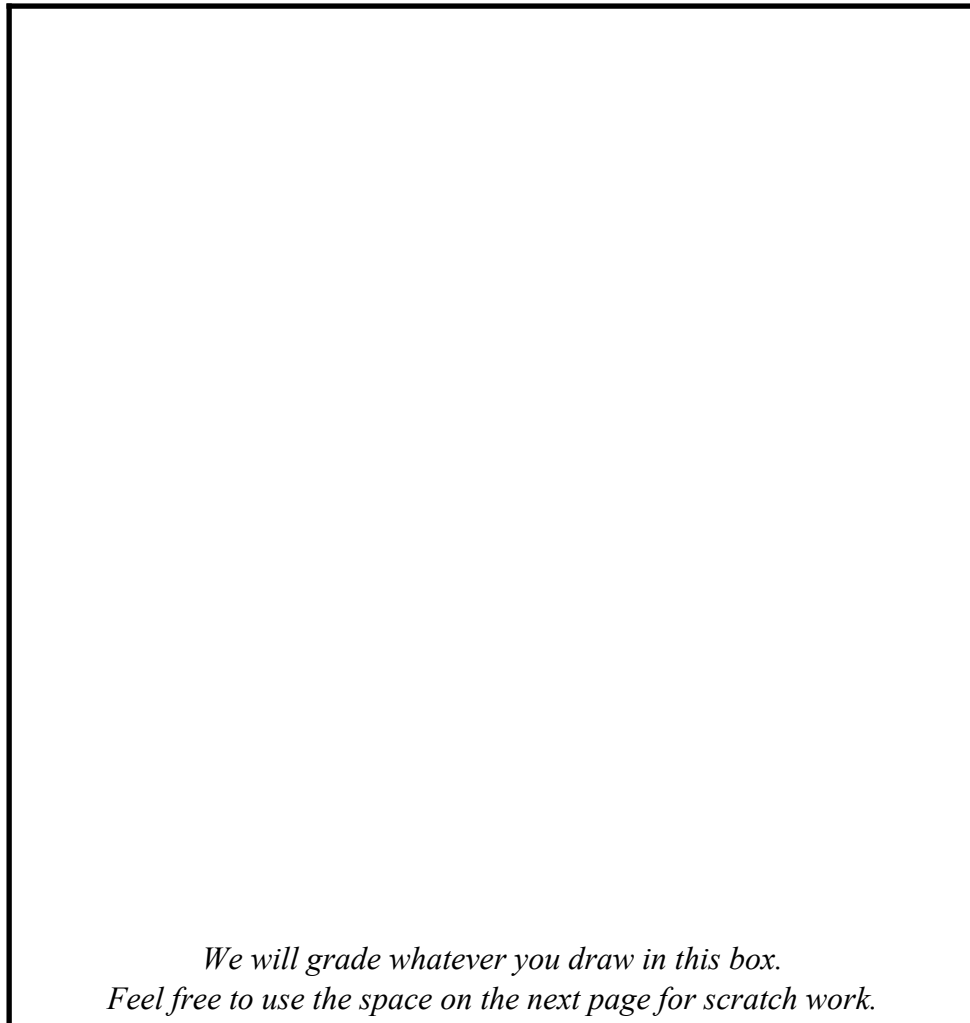
As a refresher, an undirected graph  $G = (V, E)$  is called a **friendship graph** if it satisfies the following requirement:

For any nodes  $u, v \in V$  where  $u \neq v$ ,  
there is **exactly one** node  $z \in V$  where  $\{u, z\} \in E$  and  $\{v, z\} \in E$ .

As a reminder, undirected graphs cannot have edges from nodes back to themselves.

- iii. **(3 Points)** In the space below, draw a friendship graph with exactly seven nodes. No justification is necessary.

As a hint, use the results you proved in parts (i) and (ii) to guide your search.



*We will grade whatever you draw in this box.  
Feel free to use the space on the next page for scratch work.*

**Problem Four: Induction and Cardinality****(6 Points)**

Consider the following series:

$$-1 + 2 - 3 + 4 - 5 + 6 - 7 + 8 - 9 + 10 - 11 + 12 - 13 + 14 - 15 \dots$$

We can think about evaluating larger and larger number of terms in the summation. For example, the sum of the first five terms is  $-1 + 2 - 3 + 4 - 5 = -3$ , and the sum of the first eight terms works out to  $-1 + 2 - 3 + 4 - 5 + 6 - 7 + 8 = 4$ . For notational simplicity, let's define  $A_n$  to be the sum of the first  $n$  terms in the summation. For example,  $A_0$  is the sum of the first zero terms in the summation (that's the empty sum, which is zero).  $A_1$  is the sum of the first term (-1),  $A_2$  is the sum of the first two terms ( $-1 + 2 = 1$ ),  $A_3$  is the sum of the first three terms ( $-1 + 2 - 3 = -2$ ), etc.

The following is a piecewise function that is a bijection  $f: \mathbb{N} \rightarrow \mathbb{Z}$ :

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ -\frac{n+1}{2} & \text{otherwise} \end{cases}$$

(You do not have to prove that  $f(n)$  is a bijection.) It turns out that this function is closely connected to the above series. Specifically, for every natural number  $n$ , the following is true:

$$A_n = f(n)$$

In other words, you can form a bijection from  $\mathbb{N}$  to  $\mathbb{Z}$  by considering longer and longer alternating sums of the natural numbers. Weird, isn't it?

Prove by induction on  $n$  that if  $n \in \mathbb{N}$ , then  $A_n = f(n)$ .

*(Extra space for your answer to Problem Four, if you need it.)*



The “canonical” example of a nonregular language is the language  $L_3 = \{ a^n b^n \mid n \in \mathbb{N} \}$ . It turns out that, not only is this language not regular, but most of its subsets aren't regular either.

- iii. **(3 Points)** Prove that if  $L \subseteq L_3$  and  $L$  contains infinitely many strings, then  $L$  is not regular.

In Problem Set Six, you designed a CFG for the following language:

$$ADD = \{ 1^m + 1^n \approx 1^{m+n} \mid m, n \in \mathbb{N} \}$$

Now, consider the following language over the alphabet  $\{1, +, \approx\}$ , which is a variation on  $ADD$ :

$$NEAR = \{ 1^m + 1^n \approx 1^p \mid m, n, p \in \mathbb{N} \text{ and } m + n = p + 1 \}$$

Intuitively,  $NEAR$  is the set of all arithmetic expressions where the left-hand side is exactly one greater than the right-hand side. For example:

$$111 + 1 \approx 111 \in NEAR$$

$$+ \approx \notin NEAR$$

$$11 + 111 \approx 1111 \in NEAR$$

$$1 + 1 \approx 11 \notin NEAR$$

$$1 + \approx \in NEAR$$

$$1 + 1 \approx 111 \notin NEAR$$

$$+ 1 \approx \in NEAR$$

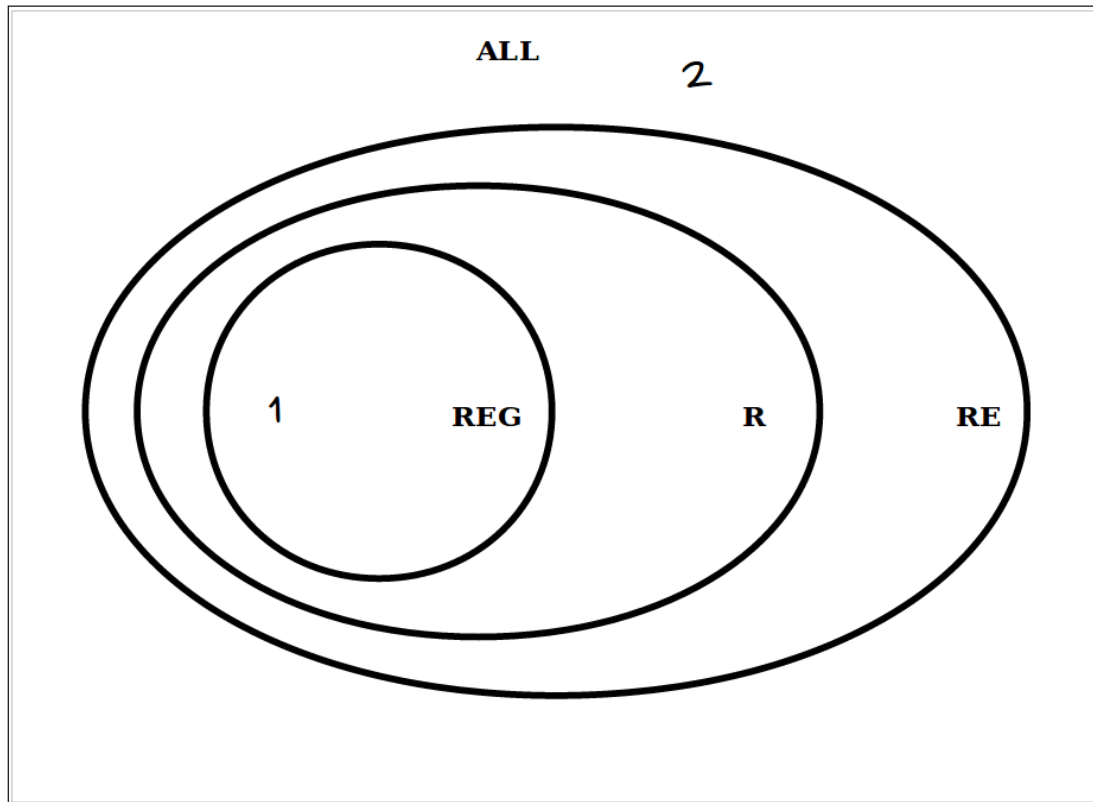
$$1 + 1 + 1 \approx 11 \notin NEAR$$

This language turns out to be context-free.

iv. **(3 Points)** Write a CFG for  $NEAR$ .

### Problem Six: R and RE Languages

**(6 Points)** Below is a Venn diagram showing the overlap of different classes of languages we've studied so far. We have also provided you a list of numbered languages. For each of those languages, draw where in the Venn diagram that language belongs. As an example, we've indicated where Language 1 and Language 2 should go. No proofs or justifications are necessary, and there is no penalty for an incorrect guess.



1.  $\Sigma^*$
2.  $L_D$
3.  $\{ w \in \{a, b\}^* \mid |w| \geq 100 \text{ and the first 50 characters of } w \text{ are the same as the last 50 characters of } w \}$
4.  $\{ \langle M_1, M_2, M_3 \rangle \mid M_1, M_2, \text{ and } M_3 \text{ are TMs over the same alphabet } \Sigma \text{ and every string in } \Sigma^* \text{ belongs to exactly one of } \mathcal{L}(M_1), \mathcal{L}(M_2), \text{ or } \mathcal{L}(M_3) \}$
5.  $HALT - A_{TM}$
6.  $A_{TM} - HALT$
7.  $\{ \langle V, w \rangle \mid V \text{ is a TM and there is a string } c \text{ such that } V \text{ accepts } \langle w, c \rangle \}$
8.  $\{ w \in \{r, d\}^* \mid w \text{ has more } r\text{'s than } d\text{'s} \}$